# Teaching Proving: From Three to Two Columns

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In teaching proofs we often present the reasoning process as a linear set of justified steps from given(s) to what is to be proved. This is most often presented as a "two-column" or Statement-Reason proof. But constructing this does not take place so linearly.

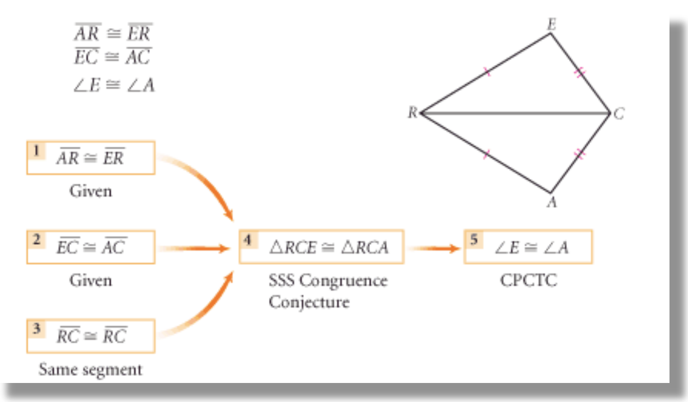
Patricio Herbst (2002) gives a detailed historical account of the development of proof in the secondary mathematics curriculum. After using the two-column format for the first time, Schultze and Sevenoak (1913) described it as follows:

> Every proof consists of a number of statements, each of which is supported by a definite reason. The only admissible reasons are: a previously proved proposition; an axiom; a definition; or the hypothesis. (p. 19)

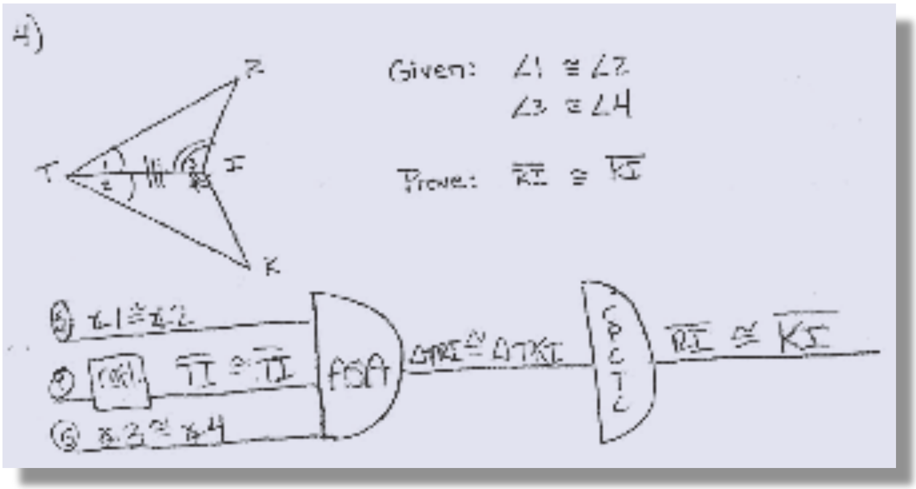
The two-column proof emphasized formalism and structure at the time it was proposed (every statement needs a reason). This linked with the idea that students were to learn to demonstrate proving, rather than develop new knowledge (exercises versus originals). Herbst argues that, "the two-column proof format brought stability to the geometry curriculum by providing a way to meld the proofs given by the text and the proofs asked from students." (p.304)
With the current emphasis (NCTM Standards, Focal Points, Common Core) on reasoning, Herbst argues for a return to knowledge construction: "Proof is essential in mathematics education not only as a valuable process for students to engage in (such as developing their capacity for mathematical reasoning) but, more importantly, as a necessary aspect of knowledge construction." (p.208)

I propose to use **instructional problem solving strategies** that allow reasoning from both the given(s) and the statement that is to be proved. In other words we are working from either end (assuming the statement that is to be proved true for the moment) to connect them somewhere in the middle. Then we retrace a path from the given(s) to the statement that is to be proved.

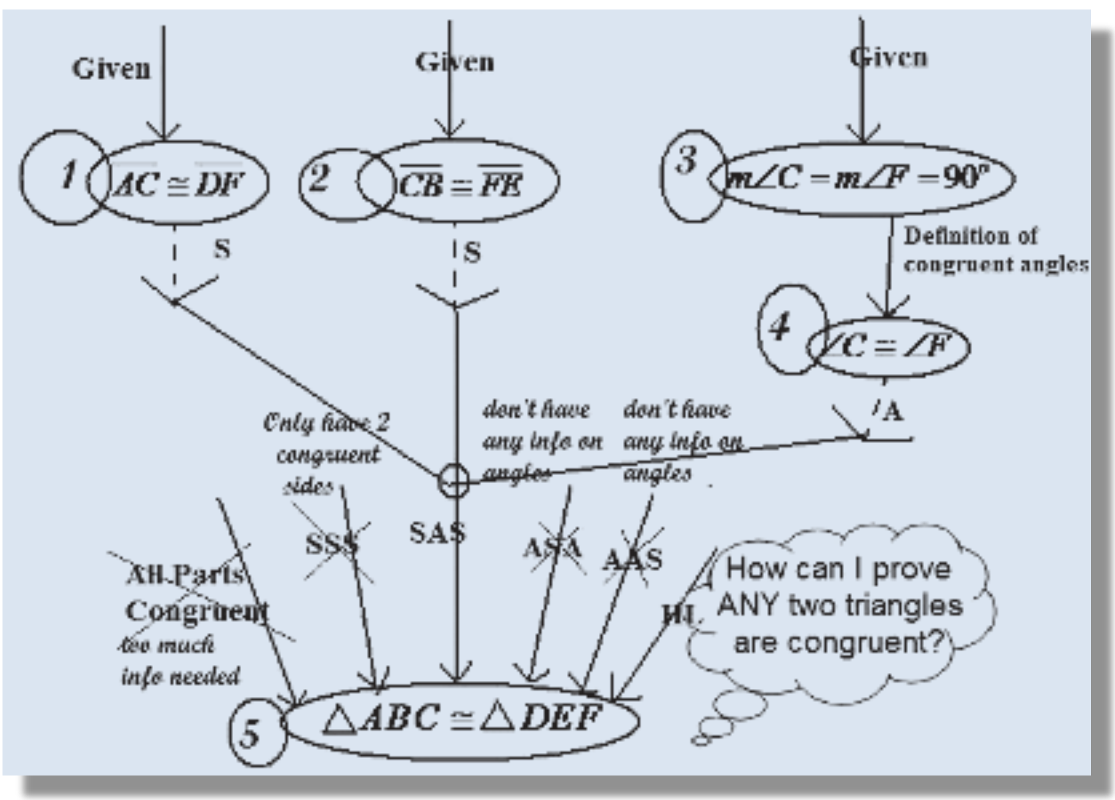
Several instructional problem solving strategies that assist students in finding such paths have been indicated in the literature and the research. More modern textbooks will use the flowchart approach, which is illustrated in the figure below:



In a recent publication Derksen, Derksen, and Cheng (2010) identified a promising approach using manipulatives and electronic circuit representations tools, whch they identified as ProofBlocks:

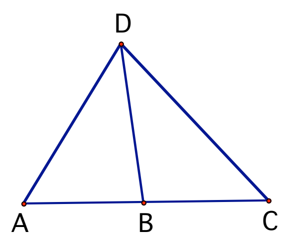



And Linares and Smith (2009) used a concept mapping approach:



All these strategies have in common that they assist students with creating a thinking structure in which the proving takes place. Each of these structures leads to a more formal presentation of the roof. Furthermore, these strategies are take the perspective of proving as a problem solving activity. The strategy that I offer with the same purpose is related to a KWL chart. KWL stands for: What do we **K**now? What do we **W**ant to know? and What have we **L**earned? It is generally organized in three columns. In doing proofs we use a similar inventory strategy as follows: What is **G**iven?, What can be **I**nferred?, What is to be **P**roved? (GIP).

Here is an example of how this might work:




In triangle ACD, B is the midpoint of $\overline{AC}$ and $\overline{AD} \cong \overline{CD}$.
Prove that $\Delta ABD \cong \Delta CBD$.

| **G**iven | **I**nferred | **P**roved |
|---|---|---|
| | Corresponding parts are congruent:<br>$\overline{AD} \cong \overline{CD}$<br>$\overline{AB} \cong \overline{BC}$<br>$\overline{BD} \cong \overline{BD}$<br>$\angle A \cong \angle C$<br>$\angle ABD \cong \angle CBD$<br>$\angle ADB \cong \angle CDB$ | $\Delta ABD \cong \Delta CBD$ |
| B is midpoint $\overline{AC}$ | definition of midpoint:<br>$\overline{AB} \cong \overline{BC}$ | |
| $\overline{AD} \cong \overline{CD}$ | Triangle ACD is isosceles.<br>Therefore, $\angle A \cong \angle C$ | |

First we begin working backwards inferring everything we can from what is to be proved. In this case we can list all corresponding parts as congruent. Then we infer what we can from the givens. Now our job is to **create a path** by selecting one or more useful inferences from what is to be proved and seek to justify it using a known property, postulate, or theorem.

In this case we select something that helps us prove two triangles congruent. We do this by working towards one of the triangle congruence theorems. In this example corresponding congruent angles may hint working toward SAS, by reasoning toward $\angle A \cong \angle C$. Or we can strive to work toward SSS by showing that $\overline{BD} \cong \overline{BD}$, by using the reflexive property. The latter seems a little easier to establish.

We are now ready to construct our proof:

Since D is the midpoint of $\overline{AC}$, it follows by definition that $\overline{AB} \cong \overline{BC}$. Because it is given that $\overline{AD} \cong \overline{CD}$, we now know that two pairs of corresponding sides of $\Delta ABD$ and $\Delta CBD$ are congruent. If we can show that the corresponding angles between these two pairs of sides are congruent or if we can show that the third pair of corresponding sides are congruent we have proved the two triangles congruent by the SAS or SSS theorem. By the reflexive property we know that $\overline{BD} \cong \overline{BD}$. Therefore the two triangles are congruent by the SSS theorem. QED.
Alternatively, $\Delta ACD$ is isosceles because $\overline{AD} \cong \overline{CD}$. Therefore $\angle A \cong \angle C$. Thus the two triangles are congruent by the SAS theorem.

For additional ideas see:

What is Proof?
http://www.cut-the-knot.org/WhatIs/WhatIsProof.shtml

Some thoughts from my good friend and brilliant teacher Mark Saul:
http://mathforum.org/~sarah/topics/proving.the.obvious.html

References:

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Derksen, J., Derksen N., and Cheng, I., (2010). ProofBlocks: A visual approach to proof. *Mathematics Teacher*, 103 (8), 571-576.

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Herbst, P.G. (2002). Engaging students in proving: A double bind on the teacher. *Journal for Research in Mathematics Education*, 33 (3), 176-203.